

# Weighted Gagliardo-Nirenberg Inequalities Involving BMO Norms and Measures

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## Abstract

Global and local weighted Gagliardo-Nirenberg inequalities with doubling measures are established. These inequalities are key ingredients for the regularity theory and existence of strong solutions for strongly coupled parabolic and elliptic systems which are degenerate or singular because of the unboundedness of dependent and independent variables.

## 1 Introduction

In [16, 17], for any  $p \geq 1$  and  $C^2$  scalar function  $u$  on  $\mathbb{R}^n$ ,  $n \geq 2$ , global and local Gagliardo-Nirenberg inequalities of the form

$$\int_{\mathbb{R}^n} |Du|^{2p+2} dx \leq C(n, p) \|u\|_{BMO}^2 \int_{\mathbb{R}^n} |Du|^{2p-2} |D^2u|^2 dx \quad (1.1)$$

were established and applied to the solvability of *scalar* elliptic equations.

More general and vectorial versions of these inequalities were presented in [8, 9] to establish the solvability of strongly coupled parabolic systems of the form *nonregular* but *uniform* parabolic system

$$\begin{cases} u_t = \operatorname{div}(A(u, Du)) + \hat{f}(u, Du) & (x, t) \in Q = \Omega \times (0, T_0), \\ u(x, 0) = U_0(x) & x \in \Omega \\ u = 0 \text{ on } \partial\Omega \times (0, T_0). \end{cases} \quad (1.2)$$

Here, and throughout this paper,  $\Omega$  is a bounded domain with smooth boundary  $\partial\Omega$  in  $\mathbb{R}^d$  for some integer  $d \geq 1$ . The temporal and  $k$ -order spatial derivatives of a vector-valued function

$$u(x, t) = (u_1(x, t), \dots, u_m(x, t))^T \quad m > 1$$

are denoted by  $u_t$  and  $D^k u$  respectively.

In this paper, we generalize global and local versions of (1.1) and the inequalities in [9] (see Corollary 2.5). Roughly speaking, we will establish inequalities of the following type: for any  $p \geq 1$  and any  $C^2$  map  $U : \Omega \rightarrow \mathbb{R}^m$

$$\int_{\Omega} \Phi^2(U) |DU|^{2p+2} d\mu \leq C \|K(U)\|_{BMO(\mu)}^2 \left\{ \int_{\Omega} \Lambda^2(U) |DU|^{2p-2} |D^2U|^2 d\mu + \dots \right\}.$$

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Here,  $K$  is a map,  $\Phi, \Lambda$  are functions on  $\mathbb{R}^m$  and  $d\mu = \omega dx$  is a doubling measure on  $\Omega$ . We assume that  $\Omega, \mu$  support a Poincaré-Sobolev type inequality.

The purpose of such generalization becomes clear when we apply the results to the study of local/global existence of strong solutions to (1.2) in our forthcoming work [10]. First of all, by replacing the Lebesgue measure  $dx$  with a general measure  $d\mu = \omega dx$ , we allow the matrices  $A, \hat{f}$  in (1.2) to depend on  $x, t$  and become degenerate or singular near a subset of  $\Omega$ .

Secondly, the degeneracy and singularity of (1.2) can also come from the behavior of the solution  $u$  itself, which is not well known as maximum principles are not available for systems (i.e.  $m > 1$ ). We replace the factor  $\|u\|_{BMO}$  in (1.1) by  $\|K(u)\|_{BMO(\mu)}$  where  $K$  is a map in  $\mathbb{R}^m$ . This allows us to deal with the case when estimates for  $\|u\|_{BMO}$ , but  $\|K(u)\|_{BMO(\mu)}$ , are not available. For example, one of the consequences of our general inequalities in this paper is the following inequality which will be useful in dealing with degenerate system in [10]: if  $\|\log(|u|)\|_{BMO(\mu)}$  is sufficiently small then

$$\int_{\Omega} |u|^{2k-2} |Du|^{2p+2} d\mu \leq C \|\log(|u|)\|_{BMO(\mu)}^2 \int_{\Omega} (|u|^{2k} |Du|^{2p-2} |D^2 u|^2 + |u|^{2k} |Du|^{2p}) d\mu.$$

Various choices of  $K$  will be discussed in [10].

We organize the paper as follows. The hypotheses and main results will be presented in Section 2. One of our key ingredients of the proof comes from Tolsa's work [14] on the  $RBMO(\mu)$  spaces and we will discuss it in Section 3. The main global and technical inequality is stated and proved in Section 4. The local version is then established in Section 5. We conclude the paper with the proof of the main inequalities and their consequences in Section 6.

## 2 Hypotheses and Main results

Throughout this paper, in our statements and proofs, we use  $C, C_1, \dots$  to denote various constants which can change from line to line but depend only on the parameters of the hypotheses in an obvious way. We will write  $C(a, b, \dots)$  when the dependence of a constant  $C$  on its parameters is needed to emphasize that  $C$  is bounded in terms of its parameters. We also write  $a \preceq b$  if there is a universal constant  $C$  such that  $a \leq Cb$ . In the same way,  $a \sim b$  means  $a \preceq b$  and  $b \preceq a$ .

For any  $\mu$ -measurable subset  $A$  of  $\Omega$  and any locally  $\mu$ -integrable function  $U : \Omega \rightarrow \mathbb{R}^m$  we denote by  $\mu(A)$  the measure of  $A$  and  $U_A$  the average of  $U$  over  $A$ . That is,

$$U_A = \fint_A U(x) d\mu = \frac{1}{\mu(A)} \int_A U(x) d\mu.$$

We say that  $\Omega$  and  $\mu$  support a Poincaré-Sobolev inequality if the following holds.

**PS)** There are  $\sigma \in (0, 1)$  and  $\tau_* \geq 1$  such that for *some*  $q > 2$  and  $q_* = \sigma q < 2$  we have

$$\frac{1}{l(B)} \left( \fint_B |u - u_B|^q d\mu \right)^{\frac{1}{q}} \leq C_{PS} \left( \fint_{\tau_* B} |Du|^{q_*} d\mu \right)^{\frac{1}{q_*}} \quad (2.1)$$

for some constant  $C_{PS}$  and any cube  $B$  with side length  $l(B)$  and any function  $u \in C^1(B)$ .

Here and throughout this paper, we write  $B_R(x)$  for a cube centered at  $x$  with side length  $R$  and sides parallel to the standard axes of  $\mathbb{R}^d$ . We will omit  $x$  in the notation  $B_R(x)$  if no ambiguity can arise. We denote by  $l(B)$  the side length of  $B$  and by  $\tau B$  the cube which is concentric with  $B$  and has side length  $\tau l(B)$ .

We have the following remark on the validity of the assumption PS).

**Remark 2.1** Suppose that  $\mu$  is doubling and supports a  $q_*$ -Poincaré inequality (see [5, eqn. (5)]): There are some constants  $C_P$ ,  $q_* \in [1, 2]$  and  $\tau_* \geq 1$  the following inequality holds true

$$\int_B |h - h_B| d\mu \leq C_P l(B) \left( \int_{\tau_* B} |Dh|^{q_*} d\mu \right)^{\frac{1}{q_*}} \quad (2.2)$$

for any cube  $B$  with side length  $l(B)$  and any function  $h \in C^1(B)$ .

Assume also that for some  $s > 0$   $\mu$  satisfies the following inequality

$$\left( \frac{r}{r_0} \right)^s \leq \frac{\mu(B_r(x))}{\mu(B_{r_0}(x_0))}, \quad (2.3)$$

where  $B_r(x), B_{r_0}(x_0)$  are any cubes with  $x \in B_{r_0}(x_0)$ . If  $q_* = 2$  then [5, Section 3] shows that a  $q_*$ -Poincaré inequality also holds for some  $q_* < 2$ . Thus, we can assume that  $q_* \in (1, 2)$ .

If  $q_* < s$  then [5, 1) of Theorem 5.1] establishes (2.1) for  $q = sq_*/(s - q_*)$ . Thus,  $q > 2$  if  $s < 2q_*/(2 - q_*)$ . This is the case if we choose  $q_* < 2$  and closed to 2. If  $s = q_*$ , [5, 2) of Theorem 5.1] shows that (2.1) holds true for any  $q > 1$ . On the other hand, if  $q_* > s$  then [5, 3) of Theorem 5.1] gives a stronger version of (2.1) for  $q = \infty$ . In particular, the Hölder norm of  $u$  is bounded in terms of  $\|Du\|_{L^{q_*}(\mu)}$ . We thus need only that  $\Omega, \mu$  support a  $q_*$ -Poincaré inequality for some  $q_* \in (1, 2)$  and (2.3) is valid for some  $s > 0$ .

To proceed, we recall some well known notions from Harmonic Analysis.

A function  $f \in L^1(\mu)$  is said to be in  $BMO(\mu)$  if

$$[f]_{*,\mu} := \sup_Q \int_Q |f - f_Q| d\mu < \infty. \quad (2.4)$$

We then define

$$\|f\|_{BMO(\mu)} := [f]_{*,\mu} + \|f\|_{L^1(\mu)}.$$

For  $\gamma \in (1, \infty)$  we say that a nonnegative locally integrable function  $w$  belongs to the class  $A_\gamma$  or  $w$  is an  $A_\gamma$  weight if the quantity

$$[w]_\gamma := \sup_{B \subset \Omega} \left( \int_B w d\mu \right) \left( \int_B w^{1-\gamma'} d\mu \right)^{\gamma-1} \text{ is finite.} \quad (2.5)$$

Here,  $\gamma' = \gamma/(\gamma - 1)$  and the supremum is taken over all cubes  $B$  in  $\Omega$ . For more details on these classes we refer the reader to [1, 15, 18].

We assume the following hypotheses.

**M)** Let  $\Omega$  be a bounded domain in  $\mathbb{R}^d$  and  $d\mu = \omega dx$  for some  $\omega \in C^1(\Omega, \mathbb{R}^+)$ . Suppose that there are a constant  $C_\mu$  and a fixed number  $n \in (0, d]$  such that : for any cube  $Q_r$  with side length  $r > 0$

$$\mu(Q_r) \leq C_\mu r^n. \quad (2.6)$$

Furthermore,  $\Omega, \mu$  satisfies PS).

**A.1)** Let  $K : \text{dom}(K) \rightarrow \mathbb{R}^m$  be a  $C^1$  map on a domain  $\text{dom}(K) \subset \mathbb{R}^m$  such that  $K_U^{-1}(U) = K_U(U)^{-1}$  exists and  $\mathbb{K}_U \in L^\infty(\text{dom}K)$ , where we will always abbreviate

$$\mathbb{K}(U) = (K_U(U)^{-1})^T. \quad (2.7)$$

Let  $\Phi, \Lambda : \text{dom}(K) \rightarrow \mathbb{R}^+$  be  $C^1$  positive functions. We assume that for all  $U \in \text{dom}(K)$

$$|\mathbb{K}(U)| \preceq \Lambda(U)\Phi^{-1}(U), \quad (2.8)$$

$$|\Phi_U(U)||\mathbb{K}(U)| \preceq \Phi(U) \text{ and } |\mathbb{K}_U(U)| \text{ is bounded.} \quad (2.9)$$

**A.2)** Let  $U : \Omega \rightarrow \text{dom}(K)$  be a  $C^2$  map such that

$$\langle \omega \Phi^2(U) \mathbb{K}(U) DU, \vec{\nu} \rangle = 0 \quad (2.10)$$

on  $\partial\Omega$  where  $\vec{\nu}$  is the outward normal vector of  $\partial\Omega$ .

**A.3)** Let  $\mathbf{W}(x) := \Lambda^{p+1}(U(x))\Phi^{-p}(U(x))$ . Assume that  $[\mathbf{W}^\alpha]_{\beta+1}$  is finite for some  $\alpha > 2/(p+2)$  and  $\beta < p/(p+2)$ .

We denote

$$I_1 := \int_{\Omega} \Phi^2(U) |DU|^{2p+2} d\mu, \quad I_2 := \int_{\Omega} \Lambda^2(U) |DU|^{2p-2} |D^2U|^2 d\mu, \quad (2.11)$$

$$\bar{I}_1 := \int_{\Omega} |\Lambda_U(U)|^2 |DU|^{2p+2} d\mu, \quad (2.12)$$

$$\check{I}_0 := \int_{\Omega} |D\omega|^2 \omega^{-2} \Lambda^2(U) |DU|^{2p} d\mu. \quad (2.13)$$

Our first main result is the following.

**Theorem 2.2** *Assume A.1)-A.3). There are constants  $C, C([\mathbf{W}^\alpha]_{\beta+1})$  for which*

$$I_1 \leq C \|K(U)\|_{BMO}^2 \left[ I_2 + \bar{I}_1 + C([\mathbf{W}^\alpha]_{\beta+1}) [I_2 + I_1 + \check{I}_0] \right]. \quad (2.14)$$

*In addition, if*

$$|\Lambda_U| \preceq \Phi. \quad (2.15)$$

*Then there is a constant  $C([\mathbf{W}^\alpha]_{\beta+1})$  such that*

$$I_1 \leq C \|K(U)\|_{BMO}^2 \left[ I_2 + I_1 + C([\mathbf{W}^\alpha]_{\beta+1}) [I_2 + I_1 + \check{I}_0] \right]. \quad (2.16)$$

*Here,  $C$  also depends on  $C_{PS}, C_\mu$ .*

Next, we have a local version of (4.10). Let  $\Omega_*$  be a subset of  $\Omega$ . In place of M), the condition on the measure  $d\mu$ , we assume that there are two functions  $\omega_*, \omega_0$  satisfying the following conditions.

**LM.0)**  $\omega_* \in C^1(\Omega)$  and satisfies  $\omega_* \equiv 1$  in  $\Omega_*$  and  $\omega_* \leq 1$  in  $\Omega$ .

$$\omega_* \equiv 1 \text{ in } \Omega_* \text{ and } \omega_* \leq 1 \text{ in } \Omega. \quad (2.17)$$

**LM.1)**  $\omega_0 \in C^1(\Omega)$  and for  $d\mu = \omega_0^2 dx$  and some  $n \in (0, d]$  we have  $\mu(B_r) \leq Cr^n$ .

**LM.2)**  $d\mu = \omega_0^2 dx$  supports the Poincaré-Sobolev inequality (2.1) in PS). In addition,  $\omega_0$  also supports a Hardy type inequality: There is a constant  $C_H$  such that for any function  $u \in C_0^1(B)$

$$\int_{\Omega} |u|^2 |D\omega_0|^2 dx \leq C_H \int_{\Omega} |Du|^2 \omega_0^2 dx \quad (2.18)$$

**Theorem 2.3** Suppose LM.0)-LM.2), A.1)-A.3) and that (compare to (2.10) with  $\omega$  being  $\omega_* \omega_0^2$ )

$$\langle \omega_* \omega_0^2 \Phi^2(U) \mathbb{K}(U) DU, \vec{\nu} \rangle = 0 \quad (2.19)$$

on  $\partial\Omega$  where  $\vec{\nu}$  is the outward normal vector of  $\partial\Omega$ .

For any  $\omega_1 \in L^1(\Omega)$  and  $\omega_1 \sim \omega_0^2$  we define  $d\mu = \omega_1 dx$  and

$$I_{1,*} := \int_{\Omega_*} \Phi^2(U) |DU|^{2p+2} d\mu, \quad (2.20)$$

$$\check{I}_{0,*} := \sup_{\Omega} |D\omega_*|^2 \int_{\Omega} \Lambda^2(U) |DU|^{2p} d\mu. \quad (2.21)$$

Then, for any  $\varepsilon > 0$  there are constants  $C, C([\mathbf{W}^\alpha]_{\beta+1})$  such that

$$I_{1,*} \leq \varepsilon I_1 + \varepsilon^{-1} C \|K(U)\|_{BMO(\mu)}^2 [I_2 + \bar{I}_1 + C([\mathbf{W}^\alpha]_{\beta+1}) [I_2 + \bar{I}_1 + \check{I}_{0,*}]]. \quad (2.22)$$

Here,  $C$  also depends on  $C_{PS}, C_\mu$  and  $C_H$ .

**Remark 2.4** A typical choice of  $\omega_0$  that satisfies the Hardy type inequality (2.18) in LM.2) is  $\omega_0(x) = d_\Omega^{\frac{\gamma}{2}}(x)$ . Then, for  $d\mu = \omega_1 dx \sim d_\Omega^\gamma dx$  we will check the conditions LM.1)-LM.2). If  $B_r$  is far away from  $\partial\Omega$ , we have  $\mu(B_r) \preceq r^d$ . Near the boundary, as  $\partial\Omega$  is  $C^1$ , we easily see that  $\mu(B_r) \preceq r^{d+\beta}$ . If  $d + \gamma \geq n$  and  $r$  is bounded then  $\mu(B_r) \leq Cr^n$ . This is the case because  $\Omega$  is bounded. Thus, for any  $\gamma > -d$ , we define  $n = \min\{d, d + \gamma\} \in (0, d]$  to see that  $\mu(B_r) \leq Cr^n$  for some constant  $C$  which is bounded in terms of  $\text{diam}(\Omega)$ .

We now recall the following Hardy inequality proved by Necas (see also the paper by Lehtbäck [11] for much more general versions)

$$\int_{\Omega} |u(x)|^q d_\Omega^{\gamma-q}(x) dx \leq C \int_{\Omega} |Du(x)|^q d_\Omega^\gamma(x) dx, \quad \gamma < q - 1. \quad (2.23)$$

We see that (2.23) in LM.2) holds true with  $q = 2, \gamma < 1$ .

An immediate consequence of Theorem 2.2 is the following main inequality in [9].

**Corollary 2.5** *Let  $U : \Omega \rightarrow \text{dom}(K)$  be a  $C^2$  vector-valued function. Suppose that either  $U$  or  $\Phi^2(u)\frac{\partial U}{\partial \nu}$  vanish on the boundary  $\partial\Omega$  of  $\Omega$ .*

*We set*

$$I_1 := \int_{\Omega} \Phi^2(U) |DU|^{2p+2} dx, \quad I_2 := \int_{\Omega} \Phi^2(U) |DU|^{2p-2} |D^2U|^2 dx, \quad (2.24)$$

$$\bar{I}_1 := \int_{\Omega} |\Phi_U(U)|^2 |DU|^{2p+2} dx. \quad (2.25)$$

*For any  $\alpha > 2/(p+2)$  and  $\beta < p/(p+2)$  we have*

$$I_1 \leq C \|U\|_{BMO(\Omega)}^2 [I_2 + \bar{I}_1 + C([\Phi^\alpha(U)]_{\beta+1})(I_2 + I_1 + \bar{I}_1)]. \quad (2.26)$$

**Proof of Corollary 2.5:** We simply choose  $\Phi = \Lambda$  and  $K(U) = U$  to see  $\mathbf{W} := \Phi$ . For  $\omega \equiv 1$ ,  $\mu$  is then the Lebesgue measure. As  $\bar{I}_1$  in (2.12) and (2.25) are the same and  $\check{I}_0 = 0$ , we then have from (2.26) from (2.14). ■

Theorem 2.3 with  $\omega_0 \equiv 1$  also implies the local version of Corollary 2.5 which is one of the key ingredients in the proof of solvability of strongly coupled parabolic systems in [9]. In this paper, we obtain a more general result with general  $\mu$  satisfying LM.0)-LM.2).

$$I_{1,*} \leq \varepsilon I_1 + \varepsilon^{-1} C \|U\|_{BMO(\Omega)}^2 [I_2 + \bar{I}_1 + C([\Phi^\alpha]_{\beta+1})[I_2 + \bar{I}_1 + \check{I}_{0,*}]]. \quad (2.27)$$

Of course, there are many ways to choose  $K, \Lambda, \Phi$  depending on different situations in applications. Let us consider another choice of  $K$  and the connection between the two terms  $\|K(U)\|_{BMO(\mu)}$ ,  $[\mathbf{W}^\alpha]_{\beta+1}$ . In this paper we will only look at the case  $K(U) = [\log(\varepsilon + |U_i|)]_{i=1}^m$ , which will be useful in dealing with porous media type parabolic systems in our forthcoming work [10]. Different choices of  $K$  will be presented in [10] too.

For  $\Lambda(U) = (\varepsilon + |U|)^k$  and  $\Phi(U) \sim |\Lambda_U(U)|$  we then define for any  $k \neq 0$  and  $\varepsilon \geq 0$

$$I_1 = \int_{\Omega} (\varepsilon + |U|)^{2k-2} |DU|^{2p+2} d\mu, \quad \check{I}_0 = \int_{\Omega} (\varepsilon + |U|)^{2k} |DU|^{2p} d\mu, \quad (2.28)$$

$$I_2 = \int_{\Omega} (\varepsilon + |U|)^{2k} |DU|^{2p-2} |D^2U|^2 d\mu. \quad (2.29)$$

**Corollary 2.6** *For  $m \geq 1$ , any  $k \neq 0$  and  $\varepsilon \geq 0$  we consider the map*

$$K(U) = [\log(\varepsilon + |U_i|)]_{i=1}^m, \quad U = [U_i]_{i=1}^m. \quad (2.30)$$

*With the notations (2.28) and (2.29) and  $\mathbf{W} = (\varepsilon + |U|)^{k+p}$ , we have*

$$I_1 \leq C \|K(U)\|_{BMO(\mu)}^2 \left[ I_2 + I_1 + C([\mathbf{W}^\alpha]_{\beta+1})[I_2 + I_1 + \check{I}_0] \right], \quad (2.31)$$

*as long as the integrals are finite. Here,  $C$  is independent of  $\varepsilon$ .*

We consider the case  $m = 1$ . As  $\mathbf{W} = \Lambda^{p+1}\Phi^{-p} = |k|^{-p}(\varepsilon + |U|)^{k+p}$ , we have  $[\mathbf{W}^\alpha]_{A_q} = [(\varepsilon + |U|)^{\alpha(k+p)}]_{A_q}$  and

$$[\log(\mathbf{W}^\alpha)]_{*,\mu} = \alpha|k + p|[\log(\varepsilon + |U|)]_{*,\mu}.$$

Via a simple use of Jensen's inequality, it is well known (e.g. see [4, Chapter 9]) that  $\|\log w\|_{BMO} \leq [w]_{A_q}$  for  $1 < q \leq 2$ . In our case,  $q = \beta + 1 < 2$  so that  $\|\log \mathbf{W}^\alpha\|_{BMO} \leq [\mathbf{W}^\alpha]_{A_q}$ . Thus, the term  $[\log(\varepsilon + |U|)]_{BMO(\mu)}$  can be controlled by  $[\mathbf{W}^\alpha]_{A_q}$ . However, this type of result is not helpful in the regularity theory of PDEs.

On the other hand, if  $\log \mathbf{W}$  is BMO then we also know that  $\mathbf{W}$  is a weight. We recall the following John-Nirenberg inequality (e.g. see [4, Chapter 9]): If  $\mu$  is doubling then for any BMO( $\mu$ ) function  $v$  there are constants  $\mathbf{c}_1, \mathbf{c}_2$ , which depend only on the doubling constant of  $\mu$ , such that

$$\int_B e^{\frac{\mathbf{c}_1}{[v]_{*,\mu}}|v-v_B|} d\mu \leq \mathbf{c}_2. \quad (2.32)$$

We then have the following result.

**Corollary 2.7** *In addition to the assumptions of Corollary 2.6 we suppose that*

$$|k + p|[\log(\varepsilon + |U|)]_{*,\mu} \leq \mathbf{c}_1\beta\alpha^{-1}. \quad (2.33)$$

*Then there is a constant  $C$ , which depends also on  $\mathbf{c}_2$ , for which*

$$I_1 \leq C\|\log(\varepsilon + |U|)\|_{BMO(\mu)}^2 \left[ I_2 + I_1 + \check{I}_0 \right]. \quad (2.34)$$

It is clear that if  $\|\log(\varepsilon + |U|)\|_{BMO(\mu)}$  is sufficiently small then (2.33) and (2.34) imply

$$I_1 \leq C\|\log(\varepsilon + |U|)\|_{BMO(\mu)}^2 \left[ I_2 + \check{I}_0 \right].$$

Of course, the above corollaries have their local versions from Theorem 2.3.

### 3 Some simple consequences from Tolsa's works

The  $RBMO(\mu)$  space was introduced by Tolsa in [13, 14]. Tolsa considered *non-doubling* measure  $\mu$  and defined

$$[f]_{*,\mu} := \sup_Q \int_{\lambda Q} |f - f_Q| d\mu \quad (3.1)$$

for some constant  $\lambda > 1$ . This constant  $\lambda$  is not important as shown in [14]. The definition of  $RBMO(\mu)$  spaces in [14] coincides with the  $BMO(\mu)$ , defined by (2.4), if  $\mu$  is doubling. It was only assumed in [14] that

**M.1)** There are a constant  $C_\mu$  and a fixed number  $n \in (0, d]$  such that for any cube  $Q_r$  with side length  $r > 0$

$$\mu(Q_r) \leq C_\mu r^n. \quad (3.2)$$

The Hardy space  $H^1(\mu)$  was introduced in [13] and the duality  $RBMO(\mu)$ - $H^1(\mu)$  was also established. For our purpose in this paper, we don't need such a full force generality and we just recall the following deep result in [14].

**Lemma 3.1** (*The Main Lemma - [14, Lemma 4.1]*) *Let  $f \in RBMO(\mu)$  with compact support and  $\int_{\Omega} f d\mu = 0$ . There exist functions  $h_m \in L^\infty(\mu)$  and  $\phi_{y;m}$ ,  $m \geq 0$ , such that*

$$f(x) = h_0(x) + \sum_{m=1}^{\infty} \int \phi_{y;m}(x) h_m(y) d\mu(y); \quad (3.3)$$

*with convergence in  $L^1(\mu)$  and*

$$\sum_{m=0}^{\infty} \|h_m\|_{L^\infty(\mu)} \leq C[f]_{*,\mu}. \quad (3.4)$$

*Importantly, the functions  $\phi_{y;m}$  satisfy the properties in Lemma 3.2 below.*

It was shown in [14] that the functions  $\phi_{y;m}$  satisfy the following properties.

**Lemma 3.2** *There is a constant  $C$ , depending also on  $C_\mu$ , such that for any  $y \in \text{supp}(\mu)$  there is some cube  $Q \subset \mathbb{R}^d$  centered at  $y$*

- 1)  $\phi_{y;m} \in C_0^1(Q)$ .
- 2)  $0 \leq \phi_{y;m}(x) \leq Cl(Q)^{-n}$  for all  $x \in Q$ .
- 3)  $|D\phi_{y;m}(x)| \leq Cl(Q)^{-n-1}$  or all  $x \in Q$ .

**Proof:** In [14, Lemma 7.8], for suitable and fixed constants  $\alpha, \beta$  and *some* cubes  $Q_1, Q_2$  concentric with  $Q$  and  $\alpha l(Q_1) \leq l(Q) \leq \beta l(Q_2)$ , 1) comes from a) of [14, Lemma 7.8] as  $\phi_{y;m} = 0$  outside  $Q_2$ . Similarly, 2) comes from [14, b) and c) of Lemma 7.8] if we note that  $l(Q) \leq |y - x|$  for  $x \in Q_2 \setminus Q_1$ . Finally, 3) comes from [14, d) of Lemma 7.8]. ■

Right after the statement of [14, Lemma 4.1], there is a short proof of the fact that the  $H^1(\mu)$  norm of a function is bounded by  $\|f\|_{L^1(\mu)} + \|M_\Phi f\|_{L^1(\mu)}$  ( $M_\Phi f$  is defined in [14, Definition 1.1] which is generally larger than the one defined in (3.6) below). For our purpose in this paper, we need only estimate  $\langle f, g \rangle$  with  $g \in RBMO(\mu)$ . We then state the following lemma.

**Lemma 3.3** *Let  $f \in RBMO(\mu)$  with the representation (3.3). Let  $F \in L^1(\mu)$  such that*

$$\int_{\Omega} F d\mu = 0, \quad M_{\hat{\Phi}} F \in L^1(\mu), \quad (3.5)$$

*where*

$$M_{\hat{\Phi}} F(y) = \sup_{m \geq 1} \int_{\Omega} \phi_{y;m}(x) F(x) d\mu(x). \quad (3.6)$$

*Then*

$$|\langle F, f \rangle| \leq C(\|F\|_{L^1(\mu)} + \|M_{\hat{\Phi}} F\|_{L^1(\mu)})[f]_{*,\mu}. \quad (3.7)$$



**Proof:** We repeat the argument right after the statement of [14, Lemma 4.1]. From (3.3), we have

$$|\langle F, f \rangle| \leq \left| \int_{\Omega} F h_0 d\mu \right| + \left| \sum_{m=1}^{\infty} \int_{\Omega} \int_{\Omega} F \phi_{y;m}(x) h_m(y) d\mu(y) d\mu(x) \right|.$$

Since

$$\left| \int_{\Omega} F \phi_{y;m}(x) h_m(y) d\mu(y) \right| \leq M_{\hat{\Phi}} F(x) \|h_m\|_{L^{\infty}(\mu)}, \quad (3.8)$$

by the definition (3.6) of  $M_{\hat{\Phi}} F$ , we have

$$|\langle F, f \rangle| \leq \|F\|_{L^1(\mu)} \|h_0\|_{L^{\infty}(\mu)} + \|M_{\hat{\Phi}} F\|_{L^1(\mu)} \sum_{m=1}^{\infty} \|h_m\|_{L^{\infty}(\mu)}.$$

By (3.4), the above gives the lemma. ■

Inspired by Lemma 3.2, we introduce the following definition.

**Definition 3.4** A function  $\phi \in C^1(\mathbb{R}^d)$  is said to be in  $\check{\Phi}$  if for any  $y \in \mathbb{R}^d$  and some cube  $Q \subset \mathbb{R}^d$  centered at  $y$  and the constant  $C$  as in Lemma 3.2

**f.1)**  $0 \leq \phi(x) \preceq Cl(Q)^{-n}$  for all  $x \in Q$ .

**f.2)**  $\phi \in C_0^1(Q)$  and  $|D\phi(x)| \preceq Cl(Q)^{-n-1}$  or all  $x \in Q$ .

For any  $F \in L^1(\mu)$  we define

$$M_{\check{\Phi}} F(y) = \sup_{\phi \in \check{\Phi}} \int_{\Omega} \phi(x) f(x) d\mu(x) \in L^1(\mu), \quad (3.9)$$

$$\|F\|_{\check{\Phi}} = \|F\|_{L^1(\mu)} + \|M_{\check{\Phi}} F\|_{L^1(\mu)}. \quad (3.10)$$

By Lemma 3.2, the functions  $\phi_{y;m}$  belong to  $\check{\Phi}$  so that  $M_{\hat{\Phi}} F(y) \leq M_{\check{\Phi}} F(y)$ . We now have from Lemma 3.3 the following result.

**Lemma 3.5** Let  $f \in BMO(\mu)$  and  $F \in L^1(\mu)$  such that

$$\int_{\Omega} F d\mu = 0. \quad (3.11)$$

Then

$$|\langle F, f \rangle| \leq C \|F\|_{\check{\Phi}} [f]_{*,\mu}. \quad (3.12)$$

We will also use the definition of the *centered* Hardy-Littlewood maximal operator acting on functions  $F \in L_{loc}^1(\mu)$

$$M(F)(y) = \sup_{\varepsilon} \left\{ \int_{B_{\varepsilon}(y)} F(x) d\mu : \varepsilon > 0 \text{ and } B_{\varepsilon}(y) \subset \Omega \right\}. \quad (3.13)$$

We also note here the Muckenhoupt theorem for non doubling measures. By [15, Theorem 3.1], we have that if  $w$  is an  $A_q(\mu)$  weight then for any  $F \in L^q(\mu)$  with  $q > 1$

$$\int_{\Omega} M(F)^q w \, d\mu \leq C(C_\mu, [w]_q) \int_{\Omega} F^q w \, d\mu. \quad (3.14)$$

In particular,

$$\int_{\Omega} M(F)^q \, d\mu \leq C_\mu \int_{\Omega} F^q \, d\mu. \quad (3.15)$$

## 4 The Main and Technical Inequality

In this section we will establish a our main global weighted Gagliardo-Nirenberg interpolation inequality. The main results stated in Section 2 are just consequences of this inequality.

Throughout this section we will always use the following notations and hypotheses. First, we repeat the condition M).

**M)** Let  $d\mu = \omega dx$  for some  $\omega \in C^1(\Omega, \mathbb{R}^+)$ . Suppose that there are a constant  $C$  and a fixed number  $n \in (0, d]$  such that : for any cube  $Q_r$  with side length  $r > 0$

$$\mu(Q_r) \leq Cr^n. \quad (4.1)$$

Furthermore,  $\Omega, \mu$  satisfies PS).

The following assumptions slightly generalize A.1)-A.3) as we do not assume (2.9) in P.1). The assumptions P.2), W) are exactly A.2), A.3).

**P.1)** Let  $K : \text{dom}(K) \rightarrow \mathbb{R}^m$  be a  $C^1$  map on a domain  $\text{dom}(K) \subset \mathbb{R}^m$  such that  $K_U(U)^{-1}$  exists for  $U \in \text{dom}(K)$ . Again, we use the notation (2.7),  $\mathbb{K}(U) = (K_U(U)^{-1})^T$ , and assume that  $\mathbb{K}_U \in L^\infty(\text{dom}K)$ .

Let  $\Phi, \Lambda : \text{dom}(K) \rightarrow \mathbb{R}^+$  be  $C^1$  positive functions. Assume that

$$|\mathbb{K}(U)| \preceq \Lambda(U) \Phi^{-1}(U) \text{ for all } U \in \text{dom}(K). \quad (4.2)$$

We also define the matrix

$$\mathbb{P}(U) := \Phi^2(U) \Lambda^{-1}(U) \mathbb{K}(U), \quad (4.3)$$

**P.2)** Let  $U : \Omega \rightarrow \text{dom}(K)$  be a  $C^2$  vector-valued function. satisfying

$$\langle \omega \Phi^2(U) \mathbb{K}(U) DU, \vec{\nu} \rangle = 0 \quad (4.4)$$

on  $\partial\Omega$ , where  $\vec{\nu}$  is the outward normal vector of  $\partial\Omega$ .

**W)** Let

$$\mathbf{W}(x) := \Lambda^{p+1}(U(x)) \Phi^{-p}(U(x)) \text{ for } x \in \Omega. \quad (4.5)$$

Asume that  $[\mathbf{W}^\alpha]_{\beta+1}$  is finite for some  $\alpha > 2/(p+2)$  and  $\beta < p/(p+2)$ .

We recall the definitions (2.11)-(2.13)

$$I_1 := \int_{\Omega} \Phi^2(U) |DU|^{2p+2} d\mu, \quad I_2 := \int_{\Omega} \Lambda^2(U) |DU|^{2p-2} |D^2U|^2 d\mu, \quad (4.6)$$

$$\bar{\mathcal{I}}_1 := \int_{\Omega} |\Lambda_U(U)|^2 |DU|^{2p+2} d\mu, \quad (4.7)$$

$$\check{\mathcal{I}}_0 := \int_{\Omega} |D\omega|^2 \omega^{-2} \Lambda^2(U) |DU|^{2p} d\mu, \quad (4.8)$$

and furthermore introduce

$$\hat{\mathcal{I}}_1 := \int_{\Omega} (|\mathbb{P}_U(U)| \Lambda(U) \Phi^{-1}(U))^2 |DU|^{2p+2} d\mu. \quad (4.9)$$

The main result of this section is the following theorem.

**Theorem 4.1** *Assume  $M$ ),  $P.1$ )- $P.2$ ) and  $W$ ). Suppose that the integrals in (4.6)-(4.8) are finite. Then there are constants  $C, C([\mathbf{W}^\alpha]_{\beta+1})$  for which*

$$I_1 \leq C \|K(U)\|_{BMO(\mu)}^2 \left[ I_2 + \bar{\mathcal{I}}_1 + C([\mathbf{W}^\alpha]_{\beta+1}) [I_2 + \hat{\mathcal{I}}_1 + \check{\mathcal{I}}_0] \right]. \quad (4.10)$$

The constant  $C$  depends on  $C_{PS}, C_\mu$  and the constant  $C$  in Definition 3.4.

The proof of this theorem will be divided into several lemmas. First of all, let  $W = K(U)$ . We then have  $DU = K_U(U)^{-1} DW$  so that, from the definition of  $\mathbb{K}(U) = (K_U^{-1})^T$ ,  $|DU|^2 = \langle \mathbb{K}(U) DU, DW \rangle$ . Hence, using the definition of  $\mathbb{P}(U) = \Phi^2(U) \Lambda^{-1}(U) \mathbb{K}(U)$  in (4.3), we can write

$$\begin{aligned} I_1 &= \int_{\Omega} \langle |DU|^{2p} \Lambda(U) \Phi^2(U) \Lambda^{-1}(U) \mathbb{K}(U) DU, DW \rangle \omega dx \\ &= \int_{\Omega} \langle |DU|^{2p} \Lambda(U) \omega \mathbb{P}(U) DU, DW \rangle dx. \end{aligned} \quad (4.11)$$

Using the boundary assumption (4.4),  $\langle \Lambda(U) \omega \mathbb{P}(U) DU, \vec{\nu} \rangle = 0$ , and applying integration by parts to the last integral, we have

$$I_1 = - \int_{\Omega} \langle \operatorname{div}(|DU|^{2p} \Lambda(U) \omega \mathbb{P}(U) DU), W \rangle dx.$$

Therefore, for  $G := \operatorname{div}(|DU|^{2p} \Lambda(U) \omega \mathbb{P}(U) DU) \omega^{-1}$

$$I_1 = - \int_{\Omega} \langle G, W \rangle d\mu. \quad (4.12)$$

From (4.4) and integrations by parts again, we see that

$$\int_{\Omega} G d\mu = \int_{\Omega} \operatorname{div}(|DU|^{2p} \Lambda(U) \omega \mathbb{P}(U) DU) dx = 0.$$

We will establish bounds for  $\|G\|_{L^1(\mu)}, \|M_{\check{\Phi}} G\|_{L^1(\mu)}$  and show that

$$\|G\|_{\check{\Phi}} \leq C \left[ I_2^{\frac{1}{2}} + \bar{I}_1^{\frac{1}{2}} + C([\mathbf{W}^\alpha]_{\beta+1})[\hat{I}_1^{\frac{1}{2}} + I_2^{\frac{1}{2}} + \check{I}_0^{\frac{1}{2}}] \right] I_1^{\frac{1}{2}}. \quad (4.13)$$

Once this is proved, we obtain from (4.12) and (3.12) of Lemma 3.5, which is applicable here by M), that  $I_1 \leq C\|K(U)\|_{RBMO(\mu)}\|G\|_{\check{\Phi}}$ . As we are assuming that  $\mu$  is doubling,  $\|K(U)\|_{RBMO(\mu)} \sim \|K(U)\|_{BMO(\mu)}$ . We then obtain

$$I_1 \leq C\|K(U)\|_{BMO(\mu)} \left[ I_2^{\frac{1}{2}} + \bar{I}_1^{\frac{1}{2}} + C([\mathbf{W}^\alpha]_{\beta+1})[\hat{I}_1^{\frac{1}{2}} + I_2^{\frac{1}{2}} + \check{I}_0^{\frac{1}{2}}] \right] I_1^{\frac{1}{2}},$$

which yields (4.10) via a simple use of Young's inequality. The proof is then complete.

To prove (4.13), we first estimate  $\|M_{\check{\Phi}} G\|_{L^1(\mu)}$  and note that

$$M_{\check{\Phi}} G = \sup_{\phi \in \check{\Phi}} \left| \int_{\Omega} \phi G \, d\mu \right| = \sup_{\phi \in \check{\Phi}} \left| \int_{\Omega} \phi g \, dx \right|$$

where

$$g := G\omega = \operatorname{div}(|DU|^{2p}\Lambda(U)\omega\mathbb{P}(U)DU). \quad (4.14)$$

Therefore, we need to establish that there are constants  $C, C([\mathbf{W}^\alpha]_{\beta+1})$  for which

$$\int_{\Omega} \sup_{\phi \in \check{\Phi}} \left| \int_{\Omega} \phi g \, dx \right| d\mu \leq C \left[ I_2^{\frac{1}{2}} + \bar{I}_1^{\frac{1}{2}} + C([\mathbf{W}^\alpha]_{\beta+1})[\hat{I}_1^{\frac{1}{2}} + I_2^{\frac{1}{2}} + \check{I}_0^{\frac{1}{2}}] \right] I_1^{\frac{1}{2}}. \quad (4.15)$$

From (4.14) we can write  $g = g_1 + g_2$  with  $g_i = \operatorname{div} V_i$ , setting

$$h := \Lambda(U)|DU|^{p-1}DU, \quad J_{0,\varepsilon} := h_{B_\varepsilon} = \fint_{B_\varepsilon} \Lambda(U)|DU|^{p-1}DU \, d\mu, \quad (4.16)$$

$$V_1 = \omega|DU|^{p+1}\mathbb{P}(U)(h - J_{0,\varepsilon}), \quad (4.17)$$

$$V_2 = \omega|DU|^{p+1}\mathbb{P}(U)J_{0,\varepsilon}. \quad (4.18)$$

We will establish (4.15) for  $g$  being  $g_1, g_2$  in the following lemmas.

In the sequel, for any  $\phi_\varepsilon \in \check{\Phi}$  and any  $y \in \mathbb{R}^d$  we denote by  $B_\varepsilon = B_\varepsilon(y)$  the corresponding cube centered at  $y$  with side length  $\varepsilon$  as in Definition 3.4.

Let us consider  $g_1$  first.

**Lemma 4.2** *There is a constant  $C$  such that*

$$\int_{\Omega} \sup_{\phi_\varepsilon \in \check{\Phi}} \left| \int_{\Omega} \phi_\varepsilon g_1 \, dx \right| d\mu \leq C \left[ I_2^{\frac{1}{2}} + \bar{I}_1^{\frac{1}{2}} \right] I_1^{\frac{1}{2}}. \quad (4.19)$$

*The constant  $C$  depends on  $C_{PS}, C_\mu$ .*

**Proof:** We use integration by parts (the boundary integral is zero because  $\phi_\varepsilon \in C_0^1(B_\varepsilon)$ ) to get

$$\begin{aligned} \left| \int_{B_\varepsilon(y)} \phi_\varepsilon(x) g_1 \, dx \right| &= \left| \int_{B_\varepsilon(y)} D\phi_\varepsilon(x) \mathbb{P}(U)(h - J_{0,\varepsilon}) |DU|^{p+1} \, d\mu \right| \\ &\leq \frac{C}{\varepsilon} \int_{B_\varepsilon(y)} |h - h_{B_\varepsilon(y)}| |\mathbb{P}(U)| |DU|^{p+1} \, d\mu. \end{aligned} \quad (4.20)$$

Here, we used the property of  $D\phi_\varepsilon$  in Definition 3.4, which states  $|D\phi_\varepsilon| \preceq \varepsilon^{-n-1}$ , and the assumption M) that  $\mu(B_\varepsilon) \preceq \varepsilon^n$ .

Note that (4.2) is equivalent to

$$|\mathbb{P}(U)| \preceq \Phi(U). \quad (4.21)$$

This and a simple use of Hölder's inequality for  $q > 2$  and (4.21) yield that the last integral in (4.20) is bounded by

$$\frac{C}{\varepsilon} \left( \int_{B_\varepsilon(y)} |h - h_{B_\varepsilon(y)}|^q \, d\mu \right)^{\frac{1}{q}} \left( \int_{B_\varepsilon(y)} [\Phi(U) |DU|^{p+1}]^{q'} \, d\mu \right)^{\frac{1}{q'}}.$$

Applying the Poincaré-Sobolev inequality (2.1) to each component of  $h$  and noting that there is a constant  $C$  such that

$$|Dh| \preceq |\Lambda_U(U)| |DU|^{p+1} + \Lambda(U) |DU|^{p-1} |D^2U|,$$

we find a constant  $C$  depends on  $C_{PS}$  such that

$$\begin{aligned} \frac{1}{\varepsilon} \left( \int_{B_\varepsilon} |h - h_{B_\varepsilon}|^q \, d\mu \right)^{\frac{1}{q}} &\leq C \left( \int_{\tau_* B_\varepsilon} |Dh|^{q_*} \, d\mu \right)^{\frac{1}{q_*}} \\ &\leq C \left[ \int_{\tau_* B_\varepsilon} (|\Lambda_U(U)|^{q_*} |DU|^{(p+1)q_*} + \Lambda^{q_*}(U) |DU|^{(p-1)q_*} |D^2U|^{q_*}) \, d\mu \right]^{\frac{1}{q_*}}. \end{aligned} \quad (4.22)$$

Using the definition of maximal functions (3.13) and combining the above estimates, we get from (4.20)

$$\sup_{\phi_\varepsilon \in \check{\Phi}} \left| \int_{\Omega} \phi_\varepsilon g_1 \, dx \right| \leq C [\Psi_1(y) + \Psi_2(y)] \Psi_3(y), \quad (4.23)$$

where  $\Psi_i(y) = (M(F_i^{q_i}(y)))^{\frac{1}{q_i}}$  with  $q_1 = q_2 = q_*$  and  $q_3 = q'$  and

$$F_1 = \Lambda_U(U) |DU|^{p+1}, \quad F_2 = \Lambda(U) |DU|^{p-1} |D^2U|, \quad F_3 = \Phi(U) |DU|^{p+1}.$$

Because  $q_i < 2$  (as  $q > 2$  and  $q_* = q\sigma < 2$ ), Muckenhoupt's inequality (3.15) implies

$$\left( \int_{\Omega} \Psi_i^2 \, d\mu \right)^{\frac{1}{2}} = \left( \int_{\Omega} M(F_i^{q_i})^{\frac{2}{q_i}} \, d\mu \right)^{\frac{1}{2}} \leq C_\mu \left( \int_{\Omega} F_i^2 \, d\mu \right)^{\frac{1}{2}}.$$

Therefore, applying Holder's inequality to (4.23) and using the above estimates and the notations (4.6) and (4.7), we obtain (4.19). ■

**Remark 4.3** We remark that (4.22) is the only place where we need the assumption PS) that  $\Omega, \mu$  support a Poincaré-Sobolev inequality.

We now turn to  $g_2$ .

**Lemma 4.4** For any  $p \geq 1$  and  $r \in (\frac{1}{p+1}, 1)$  we denote

$$\alpha(r) = \frac{r+1}{rp+r+1}, \quad \beta(r) = \frac{r(p+1)-1}{r(p+1)+1}.$$

Then there is  $C([\mathbf{W}^{\alpha(r)}]_{\beta(r)+1}) \sim [\mathbf{W}^{\alpha(r)}]_{\beta(r)+1}^{\frac{1}{\alpha(r)(p+1)}}$  such that

$$\int_{\Omega} \sup_{\phi_{\varepsilon} \in \check{\Phi}} \left| \int_{\Omega} \phi_{\varepsilon} g_2 \, dx \right| d\mu \leq C([\mathbf{W}^{\alpha(r)}]_{\beta(r)+1}) [\hat{\mathcal{I}}_1^{\frac{1}{2}} + I_2^{\frac{1}{2}} + \check{\mathcal{I}}_0^{\frac{1}{2}}] I_1^{\frac{1}{2}}. \quad (4.24)$$

**Proof:** Note that  $\operatorname{div} V_2 \leq C(J_1 + J_2 + J_3)$  for some constant  $C$  and

$$J_1 := \omega |\mathbb{P}_U| |DU|^{p+2} J_{0,\varepsilon}, \quad J_2 := \omega |\mathbb{P}(U)| |DU|^p |D^2 U| J_{0,\varepsilon},$$

$$J_3 := D\omega |\mathbb{P}(U)| |DU|^{p+1} J_{0,\varepsilon},$$

with  $J_{0,\varepsilon}$  being defined in (4.16).

Because  $\mathbb{P}(U) := \Phi^2(U) \Lambda^{-1}(U) \mathbb{K}(U)$

$$|\mathbb{P}_U| \preceq |(\Phi^2(U) \Lambda^{-1})_U(U)| |\mathbb{K}(U)| + \Phi^2(U) \Lambda^{-1}(U) |\mathbb{K}_U(U)|.$$

We thus need only that  $\mathbb{K}_U \in L^\infty(\operatorname{dom} K)$  and so does  $\mathbb{P}_U$ . Our calculations for  $J_1$  below are valid, see [2, Theorem 7.8].

In the sequel, for any  $r > 1/(p+1)$  we denote  $r^* = 1 - \frac{1}{r(p+1)}$ . We also write  $f = \Phi |DU|^{p+1}$ .

We consider  $J_{0,\varepsilon}$ . From the notation  $\mathbf{W} := \Lambda^{p+1} \Phi^{-p}$  (see (4.5))

$$J_{0,\varepsilon}(y) \leq \left| \int_{B_\varepsilon} \Lambda \Phi^{\frac{-p}{p+1}} \Phi^{\frac{p}{p+1}} |DU|^p \, d\mu \right| = \left| \int_{B_\varepsilon} \mathbf{W}^{\frac{1}{p+1}} f^{\frac{p}{p+1}} \, d\mu \right|.$$

If  $r_1 > 1/(p+1)$  we apply Hölder's inequality to the last integral to have the following estimate for  $J_{0,\varepsilon}$ .

$$J_{0,\varepsilon} \leq \left( \int_{B_\varepsilon} \mathbf{W}^{\frac{1}{r_1^*(p+1)}} \, d\mu \right)^{r_1^*} \left( \int_{B_\varepsilon} f^{pr_1} \, d\mu \right)^{\frac{1}{r_1(p+1)}}. \quad (4.25)$$

For  $J_1$ , we write  $J_1 = \omega L_* L J_{0,\varepsilon}$  with

$$L_* = |\mathbb{P}_U| \Lambda \Phi^{-1} |DU|^{p+1}, \quad L = \Lambda^{-1} \Phi |DU|.$$

By f.1) in Definition 3.4, we have  $\phi_\varepsilon(x) \preceq \varepsilon^{-n} \sim \mu(B_\varepsilon)^{-1}$  so that we can use Hölder's inequality to get for any  $s > 1$

$$\sup_{\phi_\varepsilon \in \check{\Phi}} \left| \int_{\Omega} \phi_\varepsilon J_1 \, dx \right| \leq \left( \int_{B_\varepsilon} L_*^{s'} \, d\mu \right)^{\frac{1}{s'}} \left( \int_{B_\varepsilon} L^s \, d\mu \right)^{\frac{1}{s}} J_{0,\varepsilon}.$$

We write  $L^s = \Lambda^{-s} \Phi^{\frac{-sp}{(p+1)}} \Phi^{\frac{s}{p+1}} |DU|^s$  and use Hölder's inequality to have for any  $r > 1/(p+1)$  the following estimate.

$$\left( \int_{B_\varepsilon} L^s d\mu \right)^{\frac{1}{s}} \leq \left( \int_{B_\varepsilon} |\Lambda|^{\frac{-s}{r_*}} \Phi^{\frac{sp}{r^*(p+1)}} d\mu \right)^{\frac{r^*}{s}} \left( \int_{B_\varepsilon} f^{sr} d\mu \right)^{\frac{1}{rs(p+1)}}.$$

Combining these estimates with (4.25) we then have

$$\sup_{\phi_\varepsilon \in \check{\Phi}} \left| \int_{\Omega} \phi_\varepsilon J_1 dx \right| \leq C_1 M(I_*^{s'})^{\frac{1}{s'}} M(f^{sr})^{\frac{1}{rs(p+1)}} M(f^{pr_1})^{\frac{1}{r_1(p+1)}}, \quad (4.26)$$

where, as  $\mathbf{W} := \Lambda^{p+1} \Phi^{-p}$ ,

$$\begin{aligned} C_1 &\leq \left( \int_{B_\varepsilon} \mathbf{W}^{\frac{1}{r_1^*(p+1)}} d\mu \right)^{r_1^*} \left( \int_{B_\varepsilon} |\Lambda|^{\frac{-s}{r_*}} \Phi^{\frac{sp}{r^*(p+1)}} d\mu \right)^{\frac{r^*}{s}} \\ &= \left[ \left( \int_{B_\varepsilon} \mathbf{W}^{\frac{1}{r_1^*(p+1)}} d\mu \right) \left( \int_{B_\varepsilon} \mathbf{W}^{\frac{-s}{r^*(p+1)}} d\mu \right)^{\frac{r^*}{r_1^* s}} \right]^{r_1^*}. \end{aligned}$$

We now choose  $s, r, r_1$  such that  $s' = sr = pr_1$  and  $sr < 2$ . This is the case if  $r \in (\frac{1}{p+1}, 1)$ ,  $s = (r+1)/r$  then  $s' = r+1$  and  $r_1 = (r+1)/p > 1/(p+1)$ . Let  $\alpha(r) = \frac{1}{r_1^*(p+1)}$  and  $\beta(r) = \frac{r^*}{r_1^* s}$ . With such choice of  $s, r, r_1$  we have

$$\alpha(r) = \frac{r+1}{rp+r+1}, \quad \beta(r) = \frac{r(p+1)-1}{r(p+1)+1}, \quad \alpha(r)/\beta(r) = \frac{s}{r^*(p+1)}. \quad (4.27)$$

It is clear that  $C_1 \leq C_{1,r}^{r_1^*}$  with  $r_1^* = \frac{1}{\alpha(r)(p+1)}$  and

$$C_{1,r} = \sup_{B \subset \Omega} \left( \int_B \mathbf{W}^{\alpha(r)} d\mu \right) \left( \int_B \mathbf{W}^{\frac{-\alpha(r)}{\beta(r)}} d\mu \right)^{\beta(r)}, \quad (4.28)$$

where the supremum is taken over all cubes  $B$  in  $\Omega$ . Clearly, the definition of weight (2.5) implies

$$[w]_{\nu+1} = \sup_{B \subset \Omega} \left( \int_B w d\mu \right) \left( \int_B w^{-\frac{1}{\nu}} d\mu \right)^\nu \quad \text{for all } \nu > 0. \quad (4.29)$$

From (4.29),  $C_{1,r} = [\mathbf{W}^{\alpha(r)}]_{\beta(r)+1}$ . We then have

$$C_1 \leq C_{1,r}^{r_1^*} \leq [\mathbf{W}^{\alpha(r)}]_{\beta(r)+1}^{\frac{1}{\alpha(r)(p+1)}}. \quad (4.30)$$

As  $sr = pr_1$ , we then obtain from (4.26) the following.

$$\sup_{\phi_\varepsilon \in \check{\Phi}} \left| \int_{\Omega} \phi_\varepsilon J_1 dx \right| \leq C_{1,r}^{r_1^*} M(L_*^{sr})^{\frac{1}{rs}} M(f^{sr})^{\frac{1}{rs}}.$$

Applying Hölder's inequality to the right hand side, we get

$$\int_{\Omega} \sup_{\phi_{\varepsilon} \in \check{\Phi}} \left| \int_{\Omega} \phi_{\varepsilon} J_1 \, dx \right| d\mu \preceq C_{1,r}^{r_1^*} \left( \int_{\Omega} M(L_*^{sr})^{\frac{2}{rs}} d\mu \right)^{\frac{1}{2}} \left( \int_{\Omega} M(f^{sr})^{\frac{2}{rs}} d\mu \right)^{\frac{1}{2}}.$$

Because  $q = 2/(rs) > 1$ , we can apply (3.15) to the integrals on the right and then use the definitions of  $L_*$ ,  $f$ ,  $\hat{\mathcal{I}}_1$  to see that

$$\int_{\Omega} \sup_{\phi_{\varepsilon} \in \check{\Phi}} \left| \int_{\Omega} \phi_{\varepsilon} J_1 \, dx \right| d\mu \preceq C_{1,r}^{r_1^*} \|L_*\|_2 \|f\|_2 = C(C_{1,r}) \hat{\mathcal{I}}_1^{\frac{1}{2}} I_1^{\frac{1}{2}}. \quad (4.31)$$

Next, we write  $J_2 = \omega |\mathbb{P}(U)| |DU|^{p-1} |D^2 U| |DU| J_{0,\varepsilon} = \omega L_* L J_{0,\varepsilon}$  with

$$L_* = \Lambda |DU|^{p-1} |D^2 U|, \quad L = \Lambda^{-1}(U) |\mathbb{P}(U)| |DU|.$$

We repeat the argument for  $J_1$ . Note that  $|\mathbb{P}| \leq \Phi$ , by (4.21), and therefore  $L^s \leq \Lambda^{-s} \Phi^{\frac{-sp}{(p+1)}} \Phi^{\frac{s}{p+1}} |DU|^s$ . We have the following inequality.

$$\left( \int_{B_{\varepsilon}} L^s \, d\mu \right)^{\frac{1}{s}} \leq \left( \int_{B_{\varepsilon}} |\Lambda|^{\frac{-s}{r_*}} \Phi^{\frac{sp}{r_*(p+1)}} \, d\mu \right)^{\frac{r_*}{s}} \left( \int_{B_{\varepsilon}} f^{sr} \, d\mu \right)^{\frac{1}{rs(p+1)}}.$$

The estimate (4.26) for  $J_1$  now applies to  $J_2$  and yields

$$\sup_{\phi_{\varepsilon} \in \check{\Phi}} \left| \int_{\Omega} \phi_{\varepsilon} J_2 \, dx \right| \leq C_1 M(L_*^{s'})^{\frac{1}{s'}} M(f^{sr})^{\frac{1}{rs(p+1)}} M(f^{pr_1})^{\frac{1}{r_1(p+1)}}. \quad (4.32)$$

As  $sr = pr_1$ , we have as before

$$\sup_{\phi_{\varepsilon}} \left| \int_{\Omega} \phi_{\varepsilon} J_2 \, dx \right| \preceq C_{1,r}^{r_1^*} M(L_*^{sr})^{\frac{1}{rs}} M(f^{sr})^{\frac{1}{rs}}.$$

The same argument for (4.31) for  $J_1$  with the new definition of  $L_*$  yields

$$\int_{\Omega} \sup_{\phi_{\varepsilon} \in \check{\Phi}} \left| \int_{\Omega} \phi_{\varepsilon} J_2 \, dx \right| d\mu \preceq C_{1,r}^{r_1^*} \|L_*\|_2 \|f\|_2 = C(C_{1,r}) I_2^{\frac{1}{2}} I_1^{\frac{1}{2}}. \quad (4.33)$$

Concerning  $J_3$ , we write  $J_3 = D\omega |\mathbb{P}(U)| |DU|^p |DU| J_{0,\varepsilon} = \omega L_* L J_{0,\varepsilon}$  with

$$L_* = D\omega \omega^{-1} \Lambda |DU|^p, \quad L = \Lambda^{-1} |\mathbb{P}(U)| |DU|.$$

Similar argument for  $J_2$  applying to this case then yields

$$\int_{\Omega} \sup_{\phi_{\varepsilon} \in \check{\Phi}} \left| \int_{\Omega} \phi_{\varepsilon} J_3 \, dx \right| d\mu \preceq C_{1,r}^{r_1^*} \|L_*\|_2 \|f\|_2 = C(C_{1,r}) \check{\mathcal{I}}_0^{\frac{1}{2}} I_1^{\frac{1}{2}} \quad (4.34)$$

Combining the estimates (4.31), (4.33) and (4.34), we derive (4.24). ■

Finally, we easily estimate  $\|G\|_{L^1(\mu)}$ .



**Lemma 4.5** *We have*

$$\int_{\Omega} |G| d\mu \leq CI_2^{\frac{1}{2}} I_1^{\frac{1}{2}} + C[\hat{\mathcal{I}}_1^{\frac{1}{2}} + I_2^{\frac{1}{2}} + \check{\mathcal{I}}_0^{\frac{1}{2}}] I_1^{\frac{1}{2}}.$$

**Proof:** Recall that  $G := g\omega^{-1}$  so that  $\|G\|_{L^1(\mu)} = \|g\|_{L^1(dx)}$ . We write  $g = \operatorname{div}(B|DU|^{2p}DU)$  with  $B = \omega\Lambda(U)\mathbb{P}(U)$ . First of all,

$$|\operatorname{div}(B|DU|^{2p}DU)| \leq |B_U||DU|^{2p+2} + |B||DU|^{2p}|D^2U|.$$

Because  $|B_U|$  is bounded by a multiple of

$$\{|D\omega|\omega^{-1}\Lambda(U)|\mathbb{P}(U)| + |\Lambda_U|\mathbb{P}(U)| + \Lambda(U)|\mathbb{P}_U(U)|\}|DU|^{p+1}\Lambda|Du|^p\omega,$$

we see that a simple use of Hölder's inequality as in the proof of Lemma 4.4, treating the last factor  $\omega\Lambda|Du|^p$  as  $J_0$ , implies

$$\int_{\Omega} |B_U||DU|^{2p+2} dx \leq C[\hat{\mathcal{I}}_1^{\frac{1}{2}} + I_2^{\frac{1}{2}} + \check{\mathcal{I}}_0^{\frac{1}{2}}] I_1^{\frac{1}{2}}.$$

As  $|\mathbb{P}(U)| \leq \Phi$ , we have  $|B||DU|^{2p}|D^2U| \leq \Phi|DU|^{p+1}\Lambda|DU|^{p-1}|D^2U|\omega$ . By Hölder's inequality we then obtain

$$\int_{\Omega} B|DU|^{2p+2} dx \leq CI_2^{\frac{1}{2}} I_1^{\frac{1}{2}}.$$

Combining the above estimates, we prove the lemma. ■

**Proof of Theorem 4.1:** It is now clear that the above lemmas yield

$$\|G\|_{\Phi} \leq C \left[ I_2^{\frac{1}{2}} + \check{\mathcal{I}}_1^{\frac{1}{2}} + C([\mathbf{W}^{\alpha(r)}]_{\beta(r)+1})[\hat{\mathcal{I}}_1^{\frac{1}{2}} + I_2^{\frac{1}{2}} + \check{\mathcal{I}}_0^{\frac{1}{2}}] \right] I_1^{\frac{1}{2}}. \quad (4.35)$$

Recall that  $\alpha(r) = \frac{r+1}{rp+r+1}$  and  $\beta(r) = \frac{r(p+1)-1}{r(p+1)+1}$ . We see that  $\alpha(r)$  decreases to  $2/(p+2)$  and  $\beta(r)$  increases to  $p/(p+2)$  as  $r \rightarrow 1^-$ .

From the definition of weights, a simple use of Hölder's inequality gives

$$[w^{\delta}]_{\gamma} \leq [w]_{\gamma}^{\delta} \quad \forall \delta \in (0, 1). \quad (4.36)$$

Thus, if  $\alpha > 2/(p+2)$  and  $\beta < p/(p+2)$  then for  $r$  close to 1 we have  $\alpha(r) < \alpha$  and  $\beta(r) > \beta$ . Hence, by choosing  $r$  close to 1 and using (4.36) and the open end property of weights, we see that

$$[\mathbf{W}^{\alpha(r)}]_{\beta(r)+1} \leq C[\mathbf{W}^{\alpha}]_{\beta+1}^{\frac{\alpha(r)}{\alpha}}. \quad (4.37)$$

Hence, we can replace  $C([\mathbf{W}^{\alpha(r)}]_{\beta(r)+1})$  by  $C([\mathbf{W}^{\alpha}]_{\beta+1})$  in (4.35), which yields (4.15). As  $C([\mathbf{W}^{\alpha(r)}]_{\beta(r)+1}) \sim [\mathbf{W}^{\alpha(r)}]_{\beta(r)+1}^{\frac{1}{\alpha(r)(p+1)}}$ , we can take

$$C([\mathbf{W}^{\alpha}]_{\beta+1}) \sim [\mathbf{W}^{\alpha}]_{\beta+1}^{\frac{1}{\alpha(p+1)}}.$$

As we explain earlier, (4.15) yields

$$I_1 \leq C \|K(U)\|_{BMO} \left[ I_2^{\frac{1}{2}} + \bar{I}_1^{\frac{1}{2}} + C([\mathbf{W}^\alpha]_{\beta+1}) [\hat{I}_1^{\frac{1}{2}} + I_2^{\frac{1}{2}} + \check{I}_0^{\frac{1}{2}}] \right] I_1^{\frac{1}{2}}.$$

This gives

$$I_1 \leq C \|K(U)\|_{BMO}^2 \left[ I_2 + \bar{I}_1 + C([\mathbf{W}^\alpha]_{\beta+1}) [I_2 + \hat{I}_1 + \check{I}_0] \right]. \quad (4.38)$$

The proof of the theorem is complete. ■

**Remark 4.6** The only place we use the assumption PS) is (4.22). We just need to assume that PS) holds true for  $h = \Lambda(U)|DU|^{p-1}DU$  and some measure  $\mu$  satisfying M). Combining with [5, Theorem 5.1] (see Remark 2.1), which deals only with a pair  $u, Du$ , we need only that some Poincaré's inequality (2.2), holds for the pair  $h, Dh$ . That is, we do not need (2.2) holds for any  $h$  but the function  $h = \Lambda(U)|DU|^{p-1}DU$  in the consideration.

## 5 The Local Inequality

In this section, we will establish a local version of Theorem 4.1. Let  $\Omega_*$  be a subset of  $\Omega$ . We assume that there are two functions  $\omega_*, \omega_0$  satisfying the following conditions.

**L.0)**  $\omega_* \in C_0^1(\Omega)$  and satisfies  $\omega_* \equiv 1$  in  $\Omega_*$  and  $\omega_* \leq 1$  in  $\Omega$ .

$$\omega_* \equiv 1 \text{ in } \Omega_* \text{ and } \omega_* \leq 1 \text{ in } \Omega. \quad (5.1)$$

**L.1)**  $\omega_0 \in C^1(\Omega)$  and for  $d\mu = \omega_0^2 dx$  and some  $n \in (0, d]$  we have  $\mu(B_r) \leq Cr^n$ .

**L.2)** The measure  $\omega_0^2 dx$  supports the Poincaré-Sobolev inequality (2.1) in PS). In addition,  $\omega_0$  also supports a Hardy type inequality: For any function  $u \in C_0^1(B)$

$$\int_{\Omega} |u|^2 |D\omega_0|^2 dx \leq C_H \int_{\Omega} |Du|^2 \omega_0^2 dx. \quad (5.2)$$

**Theorem 5.1** Suppose P.1)-P.2) with  $\omega = \omega_* \omega_0^2$ . Assume further that L.0)-L.2) hold true. For any  $\omega_1 \in L^1(\Omega)$  and  $\omega_1 \sim \omega_0^2$  we define  $d\mu = \omega_1 dx$  and recall the definitions (4.6)-(4.9) and introduce

$$I_{1,*} := \int_{\Omega_*} \Phi^2(U) |DU|^{2p+2} d\mu, \quad (5.3)$$

$$\check{I}_{0,*} := \sup_{\Omega} |D\omega_*|^2 \int_{\Omega} \Lambda^2(U) |DU|^{2p} d\mu. \quad (5.4)$$

Then, for any  $\varepsilon > 0$  there are constants  $C, C([\mathbf{W}^\alpha]_{\beta+1})$  such that

$$I_{1,*} \leq \varepsilon I_1 + \varepsilon^{-1} C C_*^2 [I_2 + \bar{I}_1 + C([\mathbf{W}^\alpha]_{\beta+1}) [I_2 + \hat{I}_1 + \bar{I}_1 + \check{I}_{0,*}]]. \quad (5.5)$$

Here,  $C_* := \|K(U)\|_{BMO(\mu)}$  and  $C$  depends on  $C_{PS}$ ,  $C_\mu$  and  $C_H$ .

**Proof:** We consider first the case  $\omega_1 = \omega_0^2$ . Clearly, from the definition of  $I_{1,*}$  and (4.11), we have for  $W = K(U)$

$$I_{1,*} \leq \int_{\Omega} \Phi^2(U) |DU|^{2p+2} \omega_* d\mu = \int_{\Omega} \langle |DU|^{2p} \Lambda(U) \omega_* \omega_0^2 \mathbb{P}(U) DU, DW \rangle dx.$$

As we are assuming P.2) with  $\omega = \omega_* \omega_0^2$ , (4.4) gives

$$\langle \omega_* \omega_0^2 \Phi^2(U) \mathbb{K}(U) DU, \vec{\nu} \rangle = 0 \quad (5.6)$$

on  $\partial\Omega$  where  $\vec{\nu}$  is the outward normal vector of  $\partial\Omega$ . Using this and integration by parts, we obtain

$$I_{1,*} \leq - \int_{\Omega} \langle G, W \rangle d\mu$$

for  $G := \operatorname{div}(|DU|^{2p} \Lambda(U) \omega_* \omega_0^2 \mathbb{P}(U) DU) \omega_0^{-2}$  and  $W = K(U)$ .

We now follow the proof of Theorem 4.1 to establish a similar version of (4.13), with  $d\mu = \omega_0^2 dx$ , to complete the proof. First of all, L.1) implies M.1) so that Lemma 3.5 is applicable here. We see that (4.13) holds true if (4.15) does. We then need only establish a similar version of (4.15). Again, we can write  $g = g_1 + g_2$  with  $g_i = \operatorname{div} V_i$ , setting

$$V_1 = \omega_* \omega_0^2 |DU|^{p+1} \mathbb{P}(U) (h - J_{0,\varepsilon}), \quad V_2 = \omega_* \omega_0^2 |DU|^{p+1} \mathbb{P}(U) J_{0,\varepsilon},$$

where  $h := \Lambda(U) |DU|^{p-1} DU$  and

$$J_{0,\varepsilon} := h_{B_\varepsilon} = \int_{B_\varepsilon} \Lambda(U) |DU|^{p-1} DU d\mu. \quad (5.7)$$

We revisit the lemmas giving the proof of (4.15) and estimate

$$\int_{\Omega} \sup_{\phi_\varepsilon \in \check{\Phi}} \left| \int_{\Omega} \phi_\varepsilon g_i dx \right| d\mu, \quad i = 1, 2. \quad (5.8)$$

Since  $\omega_* \leq 1$  we can discard it in the estimates for  $g_1$  after the use of integration by parts (4.20) in the proof of Lemma 4.2. Because the measure  $\mu$  supports a Poincaré-Sobolev's inequality (2.1), we can repeat the argument in the proof of Lemma 4.2 to obtain the same estimate for the integral in (5.8) with  $i = 1$ . Similarly, we drop  $\omega_*$  in  $J_i$ 's, with the exception of  $J_3$ , in the proof of Lemma 4.4 to estimate the integral in (5.8) with  $i = 2$ . Therefore,

$$\int_{\Omega} \sup_{\phi_\varepsilon \in \check{\Phi}} \left| \int_{\Omega} \phi_\varepsilon g dx \right| d\mu \leq \left[ I_2^{\frac{1}{2}} + \bar{I}_1^{\frac{1}{2}} + C([\mathbf{W}^\alpha]_{\beta+1}) [\hat{I}_1^{\frac{1}{2}} + I_2^{\frac{1}{2}} + \check{I}_*^{\frac{1}{2}}] \right] I_1^{\frac{1}{2}}.$$

Here, the term  $\check{I}_*$ , replacing  $\check{I}_0$  in (4.24), comes from the estimate for  $J_3 = D(\omega_* \omega_0^2) |\mathbb{P}(U)| |DU|^p |DU| J_{0,\varepsilon}$ . In fact, we write  $J_3 = \omega_0^2 L_* L J_{0,\varepsilon}$  for  $L_* = D(\omega_* \omega_0^2) \omega_0^{-2} \Lambda |DU|^p$  and  $L = \Lambda^{-1} |\mathbb{P}(U)| |DU|$ . We obtain the following version of (4.34) (with  $C_{1,r}$  being replaced by  $[\mathbf{W}^\alpha]_{\beta+1}$ )

$$\int_{\Omega} \sup_{\phi_\varepsilon} \left| \int_{\Omega} \phi_\varepsilon J_3 dx \right| d\mu \leq C([\mathbf{W}^\alpha]_{\beta+1}) \check{I}_*^{\frac{1}{2}} I_1^{\frac{1}{2}},$$

with  $\check{I}_* = \|L_*\|_2^2$ . That is,

$$\check{I}_* = \int_{\Omega} |D(\omega_* \omega_0^2)|^2 \omega_0^{-4} \Lambda^2 |DU|^{2p} d\mu = \int_{\Omega} |D(\omega_* \omega_0^2)|^2 \omega_0^{-2} \Lambda^2 |DU|^{2p} dx.$$

Because  $|D(\omega_* \omega_0^2)|^2 \preceq |D\omega_*|^2 \omega_0^4 + \omega_*^2 |D\omega_0|^2 \omega_0^2$ , we have

$$\check{I}_* \preceq \int_{\Omega} |D\omega_*|^2 \omega_0^2 \Lambda^2 |DU|^{2p} dx + \int_{\Omega} \omega_*^2 |D\omega_0|^2 \Lambda^2 |DU|^{2p} dx.$$

The first integral on the right hand side is less than  $\check{I}_{0,*}$ , defined by (5.4). Meanwhile, we apply the Hardy inequality (5.2) in L.2) to the second integral for  $u = \omega_* \Lambda |DU|^p$ , which belongs to  $C_0^1(\Omega)$ , and note that (as  $\omega_* \leq 1$ )

$$|Du|^2 \preceq \Lambda^2 |DU|^{2p-2} |D^2 U|^2 + |\Lambda_U|^2 |DU|^{2p} + |D\omega_*|^2 \Lambda^2 |DU|^{2p}.$$

We then have

$$\int_{\Omega} \omega_*^2 |D\omega_0|^2 \Lambda^2 |DU|^{2p} dx \leq C \int_{\Omega} |Du|^2 \omega_0^2 dx \preceq I_2 + \bar{I}_1 + \check{I}_{0,*}. \quad (5.9)$$

Thus, we get the following version of (4.15)

$$\int_{\Omega} \sup_{\phi_{\varepsilon} \in \check{\Phi}} \left| \int_{\Omega} \phi_{\varepsilon} g dx \right| d\mu \leq C \left[ I_2^{\frac{1}{2}} + \bar{I}_1^{\frac{1}{2}} + C([\mathbf{W}^{\alpha}]_{\beta+1}) [\hat{I}_1^{\frac{1}{2}} + I_2^{\frac{1}{2}} + \check{I}_{0,*}^{\frac{1}{2}}] \right] I_1^{\frac{1}{2}}.$$

The constant  $C$  depends on  $C_{PS}$ ,  $C_{\mu}$  and  $C_H$ . Similarly, Lemma 4.5 gives a similar estimate for  $\|G\|_{L^1(\mu)}$ . We then apply Lemma 3.5 as before and use Young's inequality to prove (5.5) for the case  $d\mu = \omega_0^2 dx$ .

Finally, if  $\omega_1 \sim \omega_0^2$  then the integrals in (5.5) with respect to the two measures are comparable, because  $D\omega_1, D\omega_0$  are not involved, so that (5.5) holds true as well. The proof is complete. ■

**Remark 5.2** For simplicity we assumed in L.0) that  $\omega_* \in C_0^1(\Omega)$ . More generally, we need only that  $u = \omega_* \Lambda |DU|^p \in C_0^1(\Omega)$  so that the Hardy inequality can apply in (5.9).

## 6 Proof of the Main Theorems and Further Generalizations

In this section, we present the proof of our main theorems. To begin we will state the following theorem which is an immediate consequence of the main technical result Theorem 4.1 and the definitions of the integrals in (4.6)-(4.9).

**Theorem 6.1** *Assume as in Theorem 4.1. Assume further that*

$$|\mathbb{P}_U| \Lambda \Phi^{-1} \preceq \Phi. \quad (6.1)$$

*Then there are constants  $C, C([\mathbf{W}^{\alpha}]_{\beta+1})$  for which*

$$I_1 \leq C \|K(U)\|_{BMO}^2 \left[ I_2 + \bar{I}_1 + C([\mathbf{W}^{\alpha}]_{\beta+1}) [I_2 + I_1 + \check{I}_0] \right]. \quad (6.2)$$

In addition, if

$$|\Lambda_U| \preceq \Phi \quad (6.3)$$

then

$$I_1 \leq C \|K(U)\|_{BMO}^2 \left[ I_2 + I_1 + C([\mathbf{W}^\alpha]_{\beta+1})[I_2 + I_1 + \check{I}_0] \right]. \quad (6.4)$$

**Proof:** By (6.1) and the definition of  $\hat{\mathcal{I}}_1$  in (4.9), we have  $\hat{\mathcal{I}}_1 \preceq I_1$ . Similarly, (6.3) and (4.7) give  $\tilde{\mathcal{I}}_1 \leq I_1$ . This theorem then follows from Theorem 4.1. ■

The local version Theorem 5.1 then implies the following

**Corollary 6.2** *Assume (6.1) and (6.3). Using the definitions (5.3) and (5.4) for  $I_{1,*}$  and  $\check{I}_{0,*}$ , we have*

$$I_{1,*} \leq \varepsilon I_1 + \varepsilon^{-1} C \|K(U)\|_{BMO(\Omega)}^2 [I_2 + I_1 + C([\mathbf{W}^\alpha]_{\beta+1})[I_2 + I_1 + \check{I}_{0,*}]]. \quad (6.5)$$

Concerning the condition (6.1)a and for later references, we remark the following.

**Remark 6.3** The technical theorems Theorem 4.1 and Theorem 5.1 always assume that (recalling  $\mathbb{K}(U) = (K_U^{-1}(U))^T$  and  $|\mathbb{K}(U)| = |K_U^{-1}(U)|$ )

$$|\mathbb{K}(U)| \preceq \Lambda(U) \Phi^{-1}(U). \quad (6.6)$$

The condition (6.1) can be replaced by a stronger but more verifiable one:

$$|\Phi_U(U)| |\mathbb{K}(U)| \preceq \Phi(U), \quad |\mathbb{K}_U(U)| \text{ is bounded.} \quad (6.7)$$

Indeed, from the definition  $\mathbb{P}(U) = \Phi^2(U) \Lambda^{-1}(U) \mathbb{K}(U)$ , we have

$$|\mathbb{P}_U| \Lambda \Phi^{-1} \preceq |\Phi_U| |\mathbb{K}| + \Phi |\Lambda_U| \Lambda^{-1} |\mathbb{K}| + \Phi |\mathbb{K}_U(U)|.$$

By (6.6), we have  $\Phi |\Lambda_U| \Lambda^{-1} |\mathbb{K}| \preceq |\Lambda_U|$ . Thus, if (6.7) holds then the above clearly implies  $|\mathbb{P}_U| \Lambda \Phi^{-1} \preceq \Phi + |\Lambda_U|$  so that  $\hat{\mathcal{I}}_1 \preceq I_1 + \tilde{\mathcal{I}}_1$ . Therefore, (6.2) also holds true if (6.6) and (6.7) are assumed.

**Proof of Theorem 2.2 and Theorem 2.3:** The assumption A.2) contains (6.6) and (6.7) of Remark 6.3. Therefore, under the assumptions A.1)-A.3), Theorem 2.2 follows from Theorem 6.1. In the same way, Theorem 2.3 is a consequence of Corollary 5.1. ■

Let us consider different choices of  $K$  and prove Corollary 2.6. Consider the case  $\Lambda(U) = (\varepsilon + |U|)^k$  and  $\Phi(U) \sim |\Lambda_U(U)|$  for any  $k \neq 0$  and  $\varepsilon \geq 0$ . The corresponding integrals in Theorem 6.1 are

$$I_1 = \int_{\Omega} (\varepsilon + |U|)^{2k-2} |DU|^{2p+2} d\mu, \quad \check{I}_0 = \int_{\Omega} (\varepsilon + |U|)^{2k} |DU|^{2p} d\mu, \quad (6.8)$$

$$I_2 = \int_{\Omega} (\varepsilon + |U|)^{2k} |DU|^{2p-2} |D^2U|^2 d\mu. \quad (6.9)$$

**Proof of Corollary 2.6:** We apply (6.4) of Theorem 6.1 to this case.  $\Lambda(U) = (\varepsilon + |U|)^k$ ,  $\Phi(U) = |k|(\varepsilon + |U|)^{k-1} \sim |\Lambda_U(U)|$ . It is clear that  $\mathbf{W} = \Lambda^{p+1}\Phi^{-p} \sim (\varepsilon + |U|)^{k+p}$ .

As in (2.30), we define  $K(U) = [\log(\varepsilon + |U_i|)]_{i=1}^m$ , therefore  $K_U(U) = \text{diag}[(\varepsilon + |U_i|)^{-1}]$  and  $\mathbb{K}(U) = \text{diag}[(\varepsilon + |U_i|)]$ . Hence,  $|\mathbb{K}(U)| \preceq \varepsilon + |U|$  so that  $|\mathbb{K}| \preceq \Lambda\Phi^{-1}$ . Also, it is clear that  $|\Phi_U||\mathbb{K}| \preceq \Phi$  and  $|\mathbb{K}_U(U)|$  is bounded. Hence, (6.4) holds true by Remark 6.3 and applies here to give

$$I_1 \leq C\|K(U)\|_{BMO(\mu)}^2 \left[ I_2 + I_1 + C([\mathbf{W}^\alpha]_{\beta+1})[I_2 + I_1 + \check{I}_0] \right],$$

and the proof is complete. ■

To prove Corollary 2.7 we have the following estimate for  $[\mathbf{W}^\alpha]_{\beta+1}$ .

**Lemma 6.4** *For any  $\alpha, \beta > 0$  and  $\mathbf{c}_1, \mathbf{c}_2$  as in (6.11)*

$$[\log(\mathbf{W})]_{*,\mu} \leq \mathbf{c}_1\beta\alpha^{-1} \Rightarrow [\mathbf{W}^\alpha]_{\beta+1} \leq \mathbf{c}_2^{1+\beta}. \quad (6.10)$$

**Proof:** We again recall the John-Nirenberg inequality (2.32): If  $\mu$  is doubling then for any  $BMO(\mu)$  function  $v$  there are constants  $\mathbf{c}_1, \mathbf{c}_2$ , which depend only on the doubling constant of  $\mu$ , such that

$$\int_B e^{\frac{\mathbf{c}_1}{[v]_{*,\mu}}|v-v_B|} d\mu \leq \mathbf{c}_2. \quad (6.11)$$

For any  $\beta > 0$  we know that  $e^v$  is an  $A_{\beta+1}$  weight with  $[e^v]_{\beta+1} \leq \mathbf{c}_2^{1+\beta}$  (e.g. see [4, Chapter 9]) if

$$\sup_B \int_B e^{(v-v_B)} d\mu \leq \mathbf{c}_2, \quad \sup_B \int_B e^{-\frac{1}{\beta}(v-v_B)} d\mu \leq \mathbf{c}_2. \quad (6.12)$$

It is clear that (6.12) follows from (6.11) if  $\mathbf{c}_1[v]_{*,\mu}^{-1} \geq \max\{1, \beta^{-1}\}$ . Using these facts with  $v = \alpha \log \mathbf{W}$ , we see that (6.10) holds. ■

**Proof of Corollary 2.7:** From the definition of  $\mathbf{W} = \Lambda^{p+1}\Phi^{-p} = |k|^{-p}(\varepsilon + |U|)^{k+p}$ . We then have  $[\log(\mathbf{W})]_{*,\mu} = |k + p|[\log(\varepsilon + |U|)]_{*,\mu}$ . Therefore the assumption (2.33), that  $|k + p|[\log(\varepsilon + |U|)]_{*,\mu} \leq \mathbf{c}_1\beta\alpha^{-1}$ , and (6.10) imply  $[\mathbf{W}^\alpha]_{\beta+1} \leq \mathbf{c}_2^{1+\beta}$ . The Corollary then follows from Corollary 2.6. ■

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